Exam Wistat-07

2/11/15

1. $X_1 \ldots X_n \overset{iid}{\sim} f_\theta$

$$f_\theta(x) = \begin{cases} \frac{2}{3\theta}\left(1 - \frac{x}{3\theta}\right) & 0 < x < 3\theta \\ 0 & \text{o.w} \end{cases}$$

$\boxed{\text{5 MARKS}}$

a) determine MoM of θ

$$\mu = E X$$

$$= \int_0^{3\theta} x \cdot \frac{2}{3\theta}\left(1 - \frac{x}{3\theta}\right) dx$$

$$= \left[\frac{x^2}{3\theta} - \frac{2x^3}{27\theta^2} \right]_0^{3\theta}$$

$$= \frac{(3\theta)^2}{3\theta} - \frac{(3\theta)^3}{27\theta^2}$$

$$= 3\theta - 2\theta$$

$$= \cancel{2}\,\theta$$

$$m = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Solve $\cancel{2m}$ $m = \mu$

$$\Rightarrow \boxed{\hat{\theta}_{MoM} = \frac{1}{n}\sum X_i}$$

b) Notice $\boxed{\text{5 MARKS}}$

$$
\begin{aligned}
E\hat{\theta} &= E\,\frac{1}{n}\sum_{i=1}^{n} X_i \\
&= \frac{1}{n}\sum_{i=1}^{n} EX_i \\
&= \frac{1}{n}\sum_{i=1}^{n} \theta \\
&= \theta \qquad \text{unbiased!}
\end{aligned}
$$

$$
\begin{aligned}
V(\hat{\theta}) &= V\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} V(X_i) \\
&= \frac{V(X_1)}{n}
\end{aligned}
$$

as X_1 is bounded, its variance is bounded and therefore

$$V(\hat{\theta}) \to 0 \quad \text{as} \quad n \to \infty$$

With these two results we have that $\hat{\theta}$ is consistent.

2. Cramer-Rao $x=(X_1 \dots X_n)$ $X_i \overset{iid}{\sim} f_\theta$

a. $Y = \frac{d}{d\theta} \log f_{\theta, joint}(X)$ $\boxed{\text{5 MARKS}}$

$$EY = \int \frac{d}{d\theta} \log f_{\theta, joint}(x) \cdot f_{\theta, joint}(x)\, dx$$

$$= \int \frac{f'_{\theta, joint}(x)}{\cancel{f_{\theta, joint}(x)}} \cancel{f_{\theta, joint}}(x)\, dx$$

$$= \frac{d}{d\theta} \int f_{\theta, joint}(x)\, dx$$

$$= \frac{d}{d\theta} 1$$

$$= 0$$

b) $\operatorname{Cov}(\hat{\theta}, Y) = E\,\hat{\theta} Y - E(\theta)\, EY$

$$= \int \hat{\theta}(x) \cdot \frac{f'_{\theta, joint}(x)}{f_{\cancel{\theta, joint}}(x)} f_{\theta \cancel{, joint}}(x)\, dx$$

$$= \frac{d}{d\theta} \int \hat{\theta}(x) \cdot f_{\theta, joint}(x)\, dx$$

$$= \frac{d}{d\theta} E\,\hat{\theta}(X)$$

$\boxed{\text{10 MARKS}}$

$$= \frac{d}{d\theta} \theta$$

$$= 1$$

c.) $\mathrm{Cov}(X,Y)^2 \leq V(X)\cdot V(Y)$

So

$$1 = \mathrm{Cov}(\hat{\theta}(X), Y) \leq V(\hat{\theta}(X))\ V(Y)$$

$$= V(\hat{\theta}) \cdot EY^2$$

So $V(\hat{\theta}) \geq \dfrac{1}{EY^2}$ $\boxed{\text{5 MARKS}}$

d.) as:

$$EY^2 = E\left\{\frac{d}{d\theta}\log f_{\theta,\text{joint}}(X)\right\}^2$$

$$= E\left\{\sum_{i=1}^{n}\frac{d}{d\theta}\log_{\cancel{\theta}} f_\theta(X_i)\right\}^2$$

$$= E\left\{\sum_{i,j}\frac{d}{d\theta}\log f_\theta(X_i)\,\frac{d}{d\theta}\log f_\theta(X_j)\right\}$$

$$= \sum_{i,j} E\underbrace{\left\{\frac{d}{d\theta}\log f_\theta(X_i)\,\frac{d}{d\theta}\log f_\theta(X_j)\right\}}_{\text{if } i\neq j \text{, then}}$$

expression = Cov, which is zero because X_i and X_j are indep.

$$= \sum_{i=1}^{n} E\left\{\frac{d}{d\theta}\log f_\theta(X_i)\right\}^2$$

$$= n\cdot E\left\{\frac{d}{d\theta}\log f_\theta(X_1)\right\}^2$$ $\boxed{\text{5 MARKS}}$

3. Survival regression

a) $\hat{\lambda} = \dfrac{n}{\sum_{i=1}^{n} x_i Y_i}$ $\boxed{\text{10 MARKS}}$

b) Consider joint density.

$$f_{\theta, y}(y) = \lambda^n \prod_{i=1}^{n} x_i \, e^{-\lambda \sum_{i=1}^{n} x_i y_i}$$

$\boxed{\text{5 MARKS}}$

$$= \prod_{i=1}^{n} x_i \times \underbrace{\lambda^n e^{-\lambda \frac{n}{\hat{\lambda}}}}_{\text{So } h(\lambda, \hat{\lambda})}$$

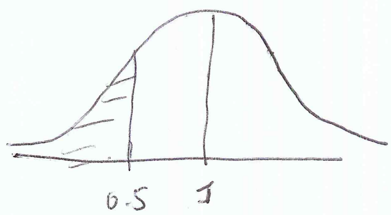
So by the factorization theorem, $\hat{\lambda}$ is sufficient

c i) Consider $\dfrac{d^2 \ell}{d\lambda^2} = -\dfrac{n}{\lambda^2}$

so $\hat{\lambda} \sim N\left(\lambda, \dfrac{\lambda^2}{n}\right)$ for large n

$H_0 : \lambda = 1$

$H_1 : \lambda \neq 1$



$$\text{p-value} = 2 \cdot P_{H_0}(\hat{\lambda} \leq 0.5)$$

$$= 2 \cdot P_{H_0}\left(\frac{\hat{\lambda} - 1}{\sqrt{1/40}} \leq \frac{0.5 - 1}{\sqrt{1/40}} \right)$$

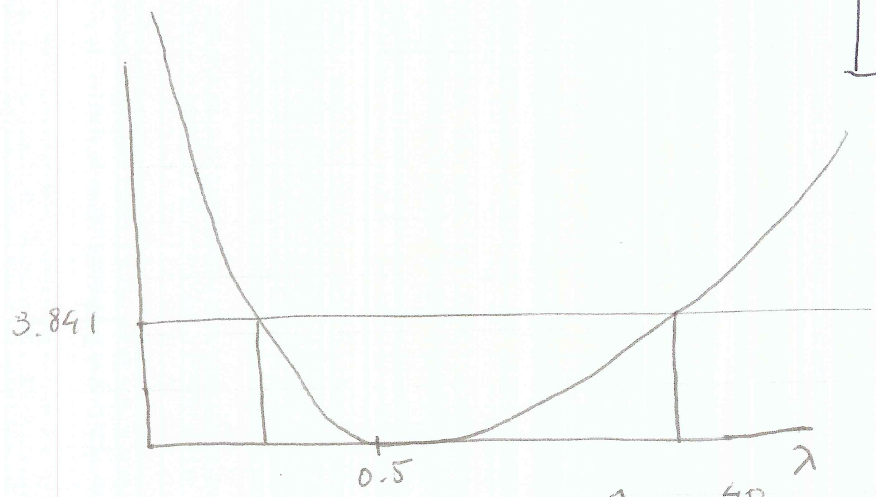
$$= 2\, P(Z \leq -3.16)$$

$$= 0.0016 < 0.05$$

So reject H_0 $\boxed{\text{10 MARKS}}$

c ii)

10 MARKS



We use $\hat{\lambda} = \frac{40}{80} = 0.5$

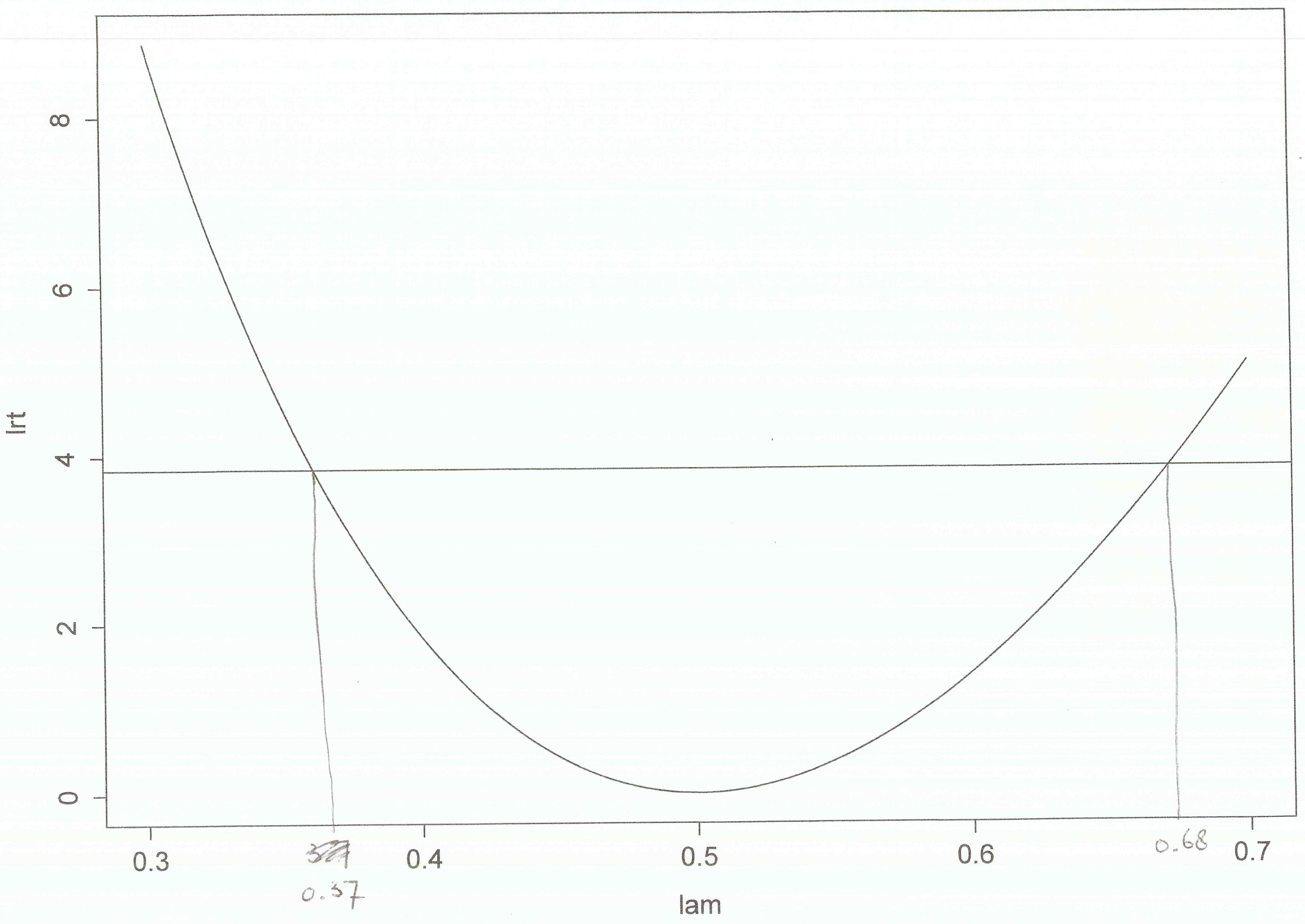
$$\Lambda = -2 \log \frac{L_Y(\theta)}{L_Y(\hat{\theta})} \Big| \theta \sim \chi^2_1 \quad \text{asymptotically}$$

note
$$\begin{aligned}\Lambda &= -2\left(\ell_Y(\theta) - \ell_Y(\hat{\theta})\right) \\ &= -2\Big\{\left(n \log \lambda + \sum_{i=1}^{n} \log x_i - \lambda \sum x_i Y_i\right) \\ &\qquad - \left(n \log \hat{\lambda} + \sum \log x_i - \hat{\lambda} \sum x_i Y_i\right)\Big\} \\ &= -2n \log \frac{\lambda}{\hat{\lambda}} + (\lambda - \hat{\lambda}) \sum_{i=1}^{n} x_i Y_i \times 2\end{aligned}$$

So $\Lambda_{observed} = 80 \log \frac{0.5}{\lambda} + (\lambda - 0.5) \cdot 80 \cdot 2$

Given the plot we find,

$(0.37, 0.68)$


8
6
4
2
0
lrt
0.3
0.4
0.5
0.6
0.7
0.37
0.68
lam

4. Optimal testing

a) X_i = i^{th} birth is boy $\quad i = 1 \ldots 1600$

$X_i \overset{iid}{\sim} \text{Bern}(p)$

$\boxed{15 \text{ MARKS}}$

$H_0: p = 0.5$

$H_1: p = 0.51$

$$CR_\alpha = \left\{ x \in \mathbb{R}^{1600} \;\middle|\; \frac{L_0(x)}{L_1(x)} \leq k_\alpha \right\}$$

$$L_0(X) = 0.5^{\sum X_i} (1-0.5)^{1600 - \sum X_i}$$

$$= 0.5^{1600}$$

$$L_1(X) = 0.51^{\sum X_i} \; 0.49^{1600 - \sum X_i}$$

$$0.05 = P(X \in CR(0.05) \mid H_0)$$

$$= P_{H_0}\left(0.5^{1600} \cdot 0.49^{1600} \cdot \left(\frac{0.51}{0.49}\right)^{-\sum X_i} \leq k_1\right)$$

$$= P_{H_0}\left(\left(\frac{0.51}{0.49}\right)^{\sum X_i} \leq k_2\right)$$

$$= P_{H_0}\Big(\underbrace{\sum X_i}_{\approx N(800,\, 400)} \geq k_3 \Big)$$

$$= P\left(Z \geq \frac{k_3 - 800}{\sqrt{400}}\right)$$

So $\dfrac{k_3 - 800}{20} = 1.645 \quad \Rightarrow \quad k_3 = 800 + 20 \cdot 1.645$

$$= 832.9$$

So $$CR(0.05) = \left\{ x \in \{0,1\}^{1600} \;\middle|\; \sum_{i=1}^{1600} x_i \geq 832.9 \right\}$$

b.) Power $= P(X \in CR \mid H_1)$

$$= P(\underbrace{\Sigma X_i}_{\approx N(816,\ 399.84)} \geq 832.9 \mid H_1)$$

$$= P\Big(Z \geq \underbrace{\frac{832.9 - 816}{\sqrt{399.84}}}_{0.845}\Big)$$

$\boxed{5 \text{ MARKS}}$

$= 0.199$